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Solutions in Riemannian space–times of arbitrary dimension

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Abstract. The Schwarzchild solution and the solution of Booth are shown to be specific examples of a genus of solutions in space–times of arbitrary spatial dimension.

1. Introduction

In a previous publication (Booth 1981) it has been shown that by considering a Riemannian geometry within a space–time of one temporal dimension and four spatial dimensions a vacuum solution exists to the extended Einstein equations

$$G_{ij}=0 \qquad (i,j=0,1,\ldots,4) \tag{1.1}$$

that correlates, on an appropriate hypersurface, with the massless Reissner–Nordstrom solution to the Einstein–Maxwell field equations. In this paper it is shown that this result and also the Schwarzchild solution are specific examples of a genus of solutions in space–times of arbitrary dimension. Furthermore, all such solutions satisfy a Birkhoff-type theorem indicating their unique, static nature.

2. The geometry

In a Riemannian space–time of one temporal dimension and n spatial dimensions that is coordinated by the cartesian set

$$\{x^0, x^1, x^2, \ldots, x^n\} \qquad (x^0=ct) \tag{2.1}$$

the 'flat-space' interval is given as

$$\mathrm{d}s^2=(\mathrm{d}x^0)^2-(\mathrm{d}x^1)^2-\ldots-(\mathrm{d}x^n)^2. \tag{2.2}$$

Defining a set of 'super-spherical'† polar coordinates

$$\{R, \chi_2, \chi_3, \ldots, \chi_n\} \tag{2.3}$$

† The phrase 'super-spherical' is somewhat inelegant. However, the analogy is clear when considering the case $n=3$ where $\chi_3=\phi$, $\chi_2=\theta$ and $R=r$—the spherical polar coordinates.

by the equations

$$x^1 = R \sin\chi_2 \sin\chi_3 \ldots \sin\chi_{n-1} \cos\chi_n,$$

$$x^2 = R \sin\chi_2 \sin\chi_3 \ldots \sin\chi_{n-1} \sin\chi_n,$$

$$x^3 = R \sin\chi_2 \sin\chi_3 \ldots \cos\chi_{n-1},$$

$$\ldots$$

$$x^n = R \cos\chi_2, \tag{2.4}$$

equation (2.2) becomes

$$ds^2 = c^2\,dt^2 - dR^2 - R^2[\![d\chi_2^2 + \sin^2\chi_2\{d\chi_3^2 + \sin^2\chi_3[d\chi_4^2 + \ldots + \sin^2\chi_{n-2} \times (d\chi_{n-1}^2 + \sin^2\chi_{n-1}\,d\chi_n^2)\ldots]\}]\!]. \tag{2.5}$$

For simplicity equation (2.5) will hereafter be written as

$$ds^2 = c^2\,dt^2 - dR^2 - R^2\,d\Omega^2. \tag{2.6}$$

3. Curved space–time

By following a simple extension of well established procedure (Weinberg 1972), it is a straightforward matter to show that the interval for a space–time distorted by a stationary, super-spherical disturbance† can be written in the form

$$ds^2 = e^{2\alpha}c^2\,dt^2 - e^{2\beta}\,dR^2 - R^2\,d\Omega^2 \tag{3.1}$$

where

$$\alpha \equiv \alpha(R, t), \qquad \beta \equiv \beta(R, t). \tag{3.2}$$

The results of analysing this metric through the Riemannian $(n+1)$ space–time and equation (1.1) are given in the Appendix, where we have listed the Christoffel symbols and Ricci tensor components. From these results it is concluded that

$$\partial\beta(R, t)/\partial t = 0 \tag{3.3}$$

whence

$$\beta \equiv \beta(R) \tag{3.4}$$

and (Landau and Lifshitz 1975)

$$\beta(R) = -\alpha(R). \tag{3.5}$$

Furthermore

$$\frac{1}{R^{n-1}}\frac{d}{dR}\left(R^{n-1}\frac{d}{dR}\right)e^{2\alpha} = 0 \tag{3.6}$$

and

$$e^{2\alpha}[2\alpha_1/R + (n-2)/R^2] = (n-2)/R^2. \tag{3.7}$$

Equation (3.6) is, remarkably, Laplace's equation in n dimensions for a single 'radial' parameter R which yields the solution

$$e^{2\alpha} = A + B/R^{n-2}. \tag{3.8}$$

† The word 'mass' only has defined meaning in $(3+1)$ space–time and accordingly is not used here.

Equation (3.7) ensures that

$$A = 1 \tag{3.9}$$

thus allowing asymptotic 'flatness'.

Clearly, for $n = 3$ this solution is the well-known Schwarzchild solution and for $n = 4$ it is the solution of Booth (1981) referred to earlier. Also Birkhoff's theorem is satisfied in respect of the unique, static nature of each solution.

4. Discussion

How delightful that Newtonian concepts of potential should be carried through for a Riemannian geometry of arbitrary spatial dimension.

The stimulus for this work is a desire to obtain a solution in (4 + 1) space–time that correlates, on the appropriate hypersurface, to the massive Reissner–Nordstrom solution. Whether this is at all possible is open to conjecture, but to this end it is noted that the Schwarzchild solution in (3 + 1) space–time is also a solution in (4 + 1) space–time.

Indeed, given the intervals

$$ds_1^2 = e^{2\alpha}c^2\, dt^2 - e^{-2\alpha}\, dR^2 - R^2\, d\Omega^2 \tag{4.1}$$

and

$$ds_2^2 = e^{2\bar{\alpha}}c^2\, dt^2 - e^{-2\bar{\alpha}}\, dR^2 - R^2\, d\Omega^2 - (dx^{n+1})^2 \tag{4.2}$$

it is found that

$$e^{2\alpha} = 1 + B/R^{n-2}. \tag{4.3}$$

Also, since

$$g_{n+1,n+1} = -1 \qquad \text{(constant)} \tag{4.4}$$

it is readily shown that

$$\bar{\alpha} = \alpha. \tag{4.5}$$

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Appendix

Here are listed the Christoffel symbols and Ricci tensor components for the metric defined by

$$ds^2 = e^{2\alpha}c^2\, dt^2 - e^{2\beta}\, dR^2 - R^2\, d\chi_2^2 - R^2 \sin^2\chi_2\, d\chi_3^2 - \dots$$
$$- R^2 \sin^2\chi_2 \sin^2\chi_3 \dots \sin^2\chi_{n-1}\, d\chi_n^2$$

where

$$\alpha \equiv \alpha(R, t), \qquad \beta \equiv \beta(R, t).$$

Christoffel symbols

$$\Gamma^0_{00} = \alpha_0, \qquad \Gamma^0_{01} = \alpha_1, \qquad \Gamma^0_{11} = \beta_0\, e^{2(\beta-\alpha)},$$

$$\Gamma^1_{00} = \alpha_1\, e^{2(\alpha-\beta)}, \qquad \Gamma^1_{01} = \beta_0, \qquad \Gamma^1_{11} = \beta_1,$$

$$\Gamma^1_{kk} = g_{kk}\, e^{-2\beta}/R \qquad (2 \leq k \leq n),$$

$$\Gamma^m_{m1} = 1/R, \qquad \Gamma^m_{mk} = \cot \chi_k \qquad (2 \leq k \leq m),$$

$$\Gamma^m_{kk} = -\cot \chi_m g^{mm} g_{kk} \qquad (m+1 \leq k \leq n-1) \quad m \geq 2$$

(no summation is implied here for repeated suffices).

Ricci tensor

$$R_{00} = e^{2(\alpha-\beta)}[\alpha_{11} + \alpha_1(n-1)/R + \alpha_1\alpha_1 - \alpha_1\beta_1] - \beta_{00} + \alpha_0\beta_0,$$

$$R_{01} = \beta_0(n-1)/R,$$

$$R_{0j} = 0, \qquad j \neq 0, 1,$$

$$R_{11} = e^{2(\beta-\alpha)}(\beta_{00} + \beta_0\beta_0 - \beta_0\alpha_0) - \alpha_{11} + \beta_1(n-1)/R - \alpha_1\alpha_1 + \beta_1\alpha_1,$$

$$R_{ij} = 0, \qquad j \neq 0, 1,$$

$$R^i_2 = \delta^i_2\left[e^{-2\beta}\left(\frac{\alpha_1}{R} - \frac{\beta_1}{R} + \frac{(n-2)}{R^2}\right) - \left(\frac{n-2}{R^2}\right)\right], \qquad i, j \geq 2.$$

In the above

$$\alpha_0 = \partial(\alpha)/\partial x^0, \qquad \alpha_1 \equiv \partial(\alpha)/\partial R, \qquad \text{etc.}$$

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